PHYC 511
Spring 2018

(1)

Problem Session 1

01/19/2018

(1) Starting from the divergence and Stoke's theorem, prove the following relations:

(i) $\int_V \vec{\nabla} \Psi \, d\tau = \oint_S \Psi \hat{n} \, da$

(ii) $\int_V \vec{\nabla} \times \vec{A} \, d\tau = \oint_S \hat{n} \times \vec{A} \, da$

(iii) $\int_S \hat{n} \times \vec{\nabla} \Psi \, da = \oint_C \Psi \, d\vec{l}$

(2) Problem 1.3, Jackson.

(3) Problem 1.5, Jackson.

(1) (i) Starting with ψ, we can define:

$\vec{A} = \psi \vec{i}$

The divergence theorem states that:

$$\int_V \vec{\nabla} \cdot \vec{A} \, d\tau = \oint_S \vec{A} \cdot \hat{n} \, da \Rightarrow \int_V \frac{\partial \psi}{\partial x} d\tau = \oint_S \psi \, n_x \, da$$

We can repeat by defining $\vec{A} = \psi \vec{j}$ and $\vec{A} = \psi \vec{k}$, resulting in:

$$\int_V \frac{\partial \psi}{\partial y} d\tau = \oint_S \psi \, n_y \, da \quad , \quad \int_V \frac{\partial \psi}{\partial z} d\tau = \oint_S \psi \, n_z \, da$$

The three relations together imply that:

$$\int_V \vec{\nabla} \psi \, d\tau = \int_V \frac{\partial \psi}{\partial x} d\tau \, \vec{i} + \int_V \frac{\partial \psi}{\partial y} d\tau \, \vec{j} + \int_V \frac{\partial \psi}{\partial z} d\tau \, \vec{k} = \oint_S \psi \, n_x \, da \, \vec{i}$$

$$+ \oint_S \psi \, n_y \, da \, \vec{j} + \oint_S \psi \, n_z \, da \, \vec{k} = \oint_S \psi \, \hat{n} \, da$$

(ii) We can apply (i) by choosing $\psi = A_x$:

$$\int_V \vec{\nabla} A_x \, d\tau = \oint_S A_x \, \hat{n} \, da \Rightarrow \int_V \frac{\partial A_x}{\partial y} d\tau = \oint_S A_x \, n_y \, da \; ,$$

$$\int \frac{\partial A_x}{\partial z} d\tau = \oint_S A_x \, n_z \, da \quad (*)$$

Similarly, for $\psi = A_y$, (i) yields:

$$\int_V \vec{\nabla} A_y \, d\tau = \oint_S A_y \hat{n} \, da \Rightarrow \int_V \frac{\partial A_y}{\partial x} d\tau = \oint_S A_y n_x \, da ,$$

$$\int_V \frac{\partial A_y}{\partial z} d\tau = \oint_S A_y n_z \, da \qquad (**)$$

Finally, for $\psi = A_z$, we find from (i):

$$\int_V \vec{\nabla} A_z \, d\tau = \oint_S A_z \hat{n} \, da \Rightarrow \int_V \frac{\partial A_z}{\partial x} d\tau = \oint_S A_z n_x \, da ,$$

$$\int_V \frac{\partial A_z}{\partial y} d\tau = \oint_S A_z n_y \, da \qquad (***)$$

By subtracting the two sides of the second relation in (**) from those of the second relation in (***), we find:

$$\int_V \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) d\tau = \oint_S (n_y A_z - n_z A_y) \, da \qquad (a)$$

From the other four relations in (*), (**), (***), we can also find:

$$\int_V \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) d\tau = \oint_S (n_z A_x - n_x A_z) \, da \qquad (b)$$

$$\int_V \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) d\tau = \oint_S (n_x A_y - n_y A_x) \, da \qquad (c)$$

Together, (a), (b), (c) give:

$$\int_V \vec{\nabla} \times \vec{A}\, d\tau = \oint_S \hat{n} \times \vec{A}\, da$$

(iii) The Stoke's theorem states that:

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, da$$

By choosing $\vec{A} = \psi \vec{i}$, this results in:

$$\oint_C \vec{A} \cdot d\vec{l} = \oint_C \psi\, dl_x$$

$$\vec{\nabla} \times \vec{A} = \frac{\partial \psi}{\partial z} \vec{j} - \frac{\partial \psi}{\partial y} \vec{k} \Rightarrow \int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, da = \int_S \left(n_y \frac{\partial \psi}{\partial z} - n_z \frac{\partial \psi}{\partial y}\right) da$$

And hence:

$$\int_S \left(n_y \frac{\partial \psi}{\partial z} - n_z \frac{\partial \psi}{\partial y}\right) da = \oint_C \psi\, dl_x \qquad (d)$$

Similarly, by choosing $\vec{A} = \psi \vec{j}$, $\vec{A} = \psi \vec{k}$, we find from Stoke's theorem:

$$\int_S \left(n_z \frac{\partial \psi}{\partial x} - n_x \frac{\partial \psi}{\partial z}\right) da = \oint_C \psi\, dl_y \qquad (e)$$

$$\int_S \left(n_x \frac{\partial \psi}{\partial y} - n_y \frac{\partial \psi}{\partial x}\right) da = \oint_C \psi\, dl_z \qquad (f)$$

Together, (d), (e), (f) give:

$$\int_S \hat{n} \times \vec{\nabla} \psi \, da = \oint_C \psi \, d\vec{l}$$

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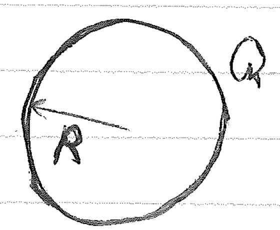
(2) (a) $\rho(\vec{x}) = \frac{Q}{4\pi R^2} \delta(r - R)$

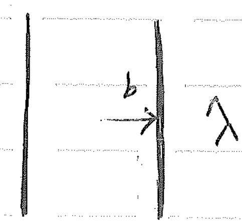


This satisfies:

integration over θ, ϕ

$$\int \rho(\vec{x}) d\tau = \int_0^\infty \frac{Q}{4\pi R^2} \delta(r-R) \, r^2 dr \times 4\pi = \boxed{Q}$$

(b) $\rho(\vec{x}) = \frac{\lambda}{2\pi b} \delta(\rho - b)$

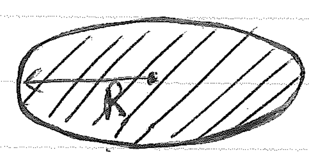


This satisfies:

integration over ϕ

$$\int \rho(\vec{x}) d\tau = \int_0^\infty \frac{\lambda}{2\pi b} \delta(\rho - b) \, \rho \, d\rho \times 2\pi = \boxed{\lambda}$$

(c) $\rho(\vec{x}) = \begin{cases} \frac{Q}{\pi R^2} \delta(z) & \rho < R \\ 0 & \rho > R \end{cases}$



This satisfies:

integration over ϕ

$$\int \rho(\vec{x}) d\tau = \int_{-\infty}^{+\infty} \frac{Q}{\pi R^2} \delta(z) \, dz \int_0^R \rho \, d\rho \times 2\pi = \boxed{Q}$$

(d) $\rho(\vec{x}) = \frac{C}{r}\,\delta\left(\theta - \frac{\pi}{2}\right)\,\Theta(R-r)$

This results in:

integration over ϕ

$$\int \rho(\vec{x})\, d\tau = C \int_0^\infty \Theta(R-r)\, r\, dr \int_0^\pi \delta\left(\theta - \frac{\pi}{2}\right) \sin\theta\, d\theta \times 2\pi = \pi R^2 C$$

needed

$$= Q \Rightarrow C = \frac{Q}{\pi R^2}$$

Thus:

$$\boxed{\rho(\vec{x}) = \frac{Q}{\pi R^2}\,\frac{1}{r}\,\delta\left(\theta - \frac{\pi}{2}\right)\,\Theta(R-r)}$$

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(3) $$\Phi = \frac{q}{4\pi\varepsilon_0}\,\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right) = \frac{q}{4\pi\varepsilon_0 r} + \frac{q}{4\pi\varepsilon_0}\left[\frac{e^{-\alpha r}-1}{r} + \frac{\alpha}{2}e^{-\alpha r}\right]$$

We have:

$$\rho(\vec{x}) = -\varepsilon_0 \nabla^2 \Phi = -\varepsilon_0 \frac{q}{4\pi\varepsilon_0}\nabla^2\left(\frac{1}{r}\right) - \varepsilon_0 \frac{q}{4\pi\varepsilon_0}\nabla^2\left[\frac{e^{-\alpha r}-1}{r} + \frac{\alpha}{2}e^{-\alpha r}\right]$$

$$= q\,\delta^{(3)}(\vec{x}) - \frac{q}{4\pi}\,\frac{1}{r}\,\frac{d^2}{dr^2}\left[r\,\frac{e^{-\alpha r}-1}{r} + \frac{\alpha r}{2}e^{-\alpha r}\right]$$

$$\nabla^2 f(r) = \frac{1}{r}\frac{d^2}{dr^2}\left(r f(r)\right)$$

Therefore:

$$\rho(\vec{x}) = q\,\delta^{(3)}(\vec{x}) - \frac{q}{4\pi r}\left(\alpha^2 e^{-\alpha r} + \frac{\alpha}{2}\cdot 2(-\alpha)\, e^{-\alpha r} + \frac{\alpha}{2}\cdot \alpha^2 r e^{-\alpha r}\right)$$

$$\boxed{= q\,\delta^{(3)}(\vec{x}) - \frac{q\alpha^3}{8\pi}\, e^{-\alpha r}}$$

The first term is the charge density of a point-like proton at the center. The second term describes the negative charge density of the electron cloud in its ground state.